# PERIODIC LONGITUDINAL FLOWS OF PSEUDOPLASTIC MATERIALS

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A model is studied in the full range of all operational parameters of the unsteady plane flow of a power-law liquid induced by periodically variable pressure drop and oscillatory motion of the walls of a plane duct. Using the theory of similarity criteria of the asymptotic behaviour are formulated in four qualitatively different rheodynamic regimes. Corresponding asymptotic expressions are found for the degree of mechanical liquidization by the action of oscillatory shear stress superimposed on the principal steady state component. Theoretical results are illustrated using a set of experimental data on the gravitational flow along a vertical oscillating sheet.

Periodic longiditunal flows (PLF), kinematically delimited by the condition of longitudinal symmetry: $v_x = 0$, $v_y = 0$, $v_z = v(t; x, y)$ and the condition of periodicity: $v(t + 2\pi/\omega; x, y) = v(t; x, y)$, may be generated in long ducts or films with a free surface by the action of periodically variable external forces. A special case of PLF are oscillatory longitudinal flows (OLF) whose time-average velocities and time-averaged external forces are zero.

PLF's have been thus far studied primarily under the simplest geometrical configurations with axial symmetry (OLF in a circular tube[1-8], general PLF's in a circular tube[9-18], general PLF's in an annulus[19]) or with plane symmetry (PLF's in flat ducts and plane films[20-22]). More complex configurations have been studied only for OLF under the linear rheodynamic regime[6].

The published papers on OLF's concentrate predominantly on the mathematically-theoretical aspects although applications to the viscometry of viscoelastic liquids exist too[3,4,7,8]. In papers[9-22] on general PLF's, in contrast, the technical aspect prevails, associated with the possibility of mechanical liquidization of non-Newtonian materials[23-27]. Under the term of mechanical liquidization we shall understand increased mean shear rate for a given mean shear stress or decreased mean shear stress for a given mean shear rate as a consequence of the superimposed oscillatory shearing component.

The behaviour of rheologically complex materials undergoing PLF is a result of three physical factors: the character of the external periodic driving forces (steady versus the oscillatory component), rheological properties of the material and the inertia of the material (affecting the mechanism of propagation of the shearing oscillations throughout the material).

*Rheological effects during* PLF are fully represented by a scalar constitutive functional expressing the response of the material in terms of the shear stress to the deformation history under the conditions of the unsteady viscometric flow[28]. It is convenient to extract from this complex description of the stress response the purely viscous response represented by the viscosity function of the steady viscometric flow and view the "rest" as a manifestation of time-dependent rheological properties (elasticity, thixotropy). Under PLF one can observe also wall effects as a consequence of induced inhomogeneity of the material, which though has not been studied quantitatively.

For purely viscous materials the susceptibility to mechanical liquidization is quantitatively represented by the viscosity function[9,10,12–17,20–22,27]. For viscoelastic materials[11,18,19,24,26] individual rheological effects cannot be separated as a matter of principle on the one hand, and, on the other hand, because the material constants appear both in the representation of the viscosity function and the time-dependent properties. Experimental results on mechanical liquidization during PLF (ref.[9,13,16,20,21,27]), however, strongly suggest that the principal constitutive cause of the ability for mechanical liquidization is pseudoplasticity (the decrease of apparent viscosity with increasing shear stress under the conditions of the steady viscometric flow) of the tested polymeric systems. Viscoelastic time-dependent effects seem to have only secondary importance[9,11,18,20,26]. As a guide for the assessment of possible influence of time-dependent rheological phenomena on the degree of liquidization may serve the fact that for any linear constitutive model of viscoelastic behaviour the superimposed oscillatory components are without effect on the mean shear rate and stress.

Another factor controlling the character of PLF and the degree of liquidization is the ratio of the oscillatory and the steady state component of the external forces of the process. This ratio may be generally characterized through the amplitude of the oscillatory component and the steady component of the shear stress on the wall of the duct, $C = \tau_0/\tau_s$. It is obvious that for $C \ll 1$ the shearing oscillation shall be only secondary as far as the time-averaged characteristics of the flow are concerned, while for $C \gg 1$ the effect of the superimposed oscillations may become dramatic. The majority of theoretical papers on viscoelastic PLF's are confined to the region of low values of $C$ when the net effect of liquidization is not conspicuous (increased flow rate by 10 to 50%). This restriction in papers based on nonlinear viscoelastic models[11,18,19,26] stems from the applied analytical method of asymptotic expansions with respect to a small parameter identified with the ratio of the oscillatory and steady state component of stress.

The region of intermediate of the ratio $\tau_0/\tau_s$ has been explored thus far for the models of purely viscous behaviour[10,12,13,15,16]. In region of extremely high values of the $\tau_0/\tau_s$ ratio one can encounter a situation when the field of the oscillatory stress or the velocity field develops practically independently of the steady state component of the flow, *i.e.* becomes identical with the field of corresponding OLF's. Mechanical liquidization thus may be modelled by the field of apparent viscosities, independent of the steady state component of the flow[25,29,30].

The extent of inertia effects strongly affects the degree of mechanical liquidization for the given periodic external forces. Principally one can distinguish creeping regimes of PLF's, where the effect of inertia as compared to the deformation forces is negligible, and, boundary layer regimes where the inertia effects are dominant. Under the creeping flow regime the effect of a given oscillatory component becomes manifest most[10,12,13,21]. Under the boundary layer regime the oscillatory component of the shear stresses is significant only within a thin shell of the material of thickness $\delta$ in the close proximity of the wall, *i.e.* in region of the oscillatory boundary layer[2,6,31].

Shearing oscillations in the material undergoing PLF may be generated by various technically plausible means: *a*) Relative oscillatory motion of the walls with respect to one another. This way[19] has been utilized predominantly in the rheometry[26], although it may acquire also some technical use[19]. *b*) Forced oscillations of the volume flow rate generated by piston pumps[23,24]. *c*) Oscillatory motion of the walls of the duct as a whole with respect to an inertial frame of reference (vibrating flows)[15,16,20–22,31]. *d*) Oscillations of pressure drop with respect to an inertial frame of reference (pulsating flows)[1–18].

It is convenient to distinguish between kinematically generated PLF's (groups (*a*), (*b*) and corresponding combinations), and dynamically generated PLF's (groups (*c*), (*d*) and corresponding combinations), for these two groups differ one from another both rheodynamically and from the application point of view.

The greatest attention so far have attracted pulsating flows in a tube[1–18]. In spite of the proclaimed technical applications[23,24] motivated by presumed energy savings during pulsation pumping of liquids, these papers remain largely academic. Aside from the unfounded concept that a superimposed oscillatory component of the stress should lead to reduced mechanical energy input, such a possibility would encounter also considerable technical difficulties associated with the purposely generated pressure shocks in a long pipe. In technical practice, on the contrary, ways are sought of subduing pressure pulsations generated by the action of piston pumps. The theory of pulsating viscoelastic flows thus would find a better use in design of necessary shock absorbers.

From the technical point of view as far more attractive appears vibrating PLF where the oscillatory component of the stress is generated by the motion of the walls of the equipment with the aim to achieve a required flow rate for an apriori selected level of driving forces. By the vibrational liquidization one can often achieve technically significant gravitational flow of concentrated suspensions of viscoplastic character in packed equipment for contacting with gas, in siphon overflows, filtration, *etc.* Of importance for scanning and control of the flow of suspensions may be also the fact that the relation between the time-averaged volume flow rate and the time-averaged pressure drop in a vibrating duct approaches a linear dependence. Vibrational flows, unlike pulsating flows, has not been studied theoretically sufficiently extensively[15,16,21,22]. Typical for their application is the relatively broad interval of operational parameters (cross section of the duct, amplitude and frequency of oscillations of the wall).

In this study an attempt is made to formulate a qualitative rheodynamic theory of all dynamically generated PLF's (pulsating, vibrating and combined), between which an exact analogy exists[32]. Attention shall be focused on the observation of the rheodynamics of PLF's in the whole range of operational parameters which should lead to the formulation of asymptotic theories. In this effort we shall confine ourselves to purely viscous non-Newtonian materials represented by the power law model of the viscosity function. In view of the fact that technical applications are perspective mainly in case of nonelastic highly viscous suspensions and because a more profound effect of viscoelastic properties on the degree of fluidization has not been proven to date, such an approach seems adequate. The power-law representation of the viscosity function substantially simplified the theoretical analysis of the problem from the viewpoint of the theory of similarity and with a cautious application of the mathematical model remains also adequate for the description of the real behaviour of purely viscous materials.

For the sake of simplicity we shall confine ourselves to the case of PLF with plane symmetry which though possesses important applications in connection with the film flow and contacting of highly viscous non-Newtonian materials with gases (absorption, dehydration, demonomerization[20–22]).

*Formulation of the Problem*

We shall be dealing with periodic longitudinal flows (PLF) of the power law liquid, generated dynamically by harmonic variations of external effects. We shall confine ourselves to cases with plane symmetry, *i.e.* to the flows in an infinite plane duct confined by walls in the planes $x = 0$ and $x = 2h$, and, to film flows on a plane wall at $x = 0$ with plane free surface at $x = h$. In an inertial frame of reference (primed quantities) the appropriate equation of motion may be written as

$$\varrho\, \partial'_t v' = \varrho g'_z - \partial'_z p' + \partial'_x \tau'_{zx} \tag{1}$$

where

$$\partial'_z p' = P'_s - P'_0 \sin(\omega t')\,. \tag{2}$$

The walls of the duct perform a longitudinal oscillatory motion with an instantaneous velocity

$$v'_w = a'\omega \cos(\omega t' - \varphi_0)\,. \tag{3}$$

This PLF shall be examined in a new reference system[32] (nonprimed quantities), performing with respect to the original inertial frame such a harmonic oscillatory motion that the apparent acceleration forces in the new system just balance the oscillatory component of pressure drop. With a suitable change of time origin the harmonically oscillating velocity of the duct walls may be expressed with respect to the new frame of reference as

$$v_w(t) = a\omega \cos(\omega t)\,, \tag{4}$$

where

$$a = ((a' \cos \varphi_0 + P'_0/(\varrho\omega^2))^2 + (a' \sin \varphi_0)^2)^{1/2}\,. \tag{5}$$

The equation of motion (*1*) takes in the new coordinates the form

$$\varrho\, \partial_t v - \varrho g_z = \partial_x \tau_{xz}\,, \tag{6}$$

where the parameter $g_z$ incorporates the effects of all steady external forces, gravity and pressure drop

$$g_z = g'_z - P'_s/\varrho\,. \tag{7}$$

Since the inertial and the canonic frame of reference have a common mean position the resultant time-averaged volume flow rate in both systems is the same. The objective quantities[28] (rates of deformation, stresses, dissipation) are clearly identical in both systems. The mathematical model of the considered process in terms of the velocity field $v_z = v(t, x)$ is represented by the equation of motion (6), the constitutive model

$$\tau_{zx} = \tau = K[\partial_z v]^n \quad (8)$$

and the boundary conditions. These determine the velocity on the oscillating wall according to Eq. (4) as

$$v = a\omega \cos(\omega t)\,; \quad x = 0 \quad (9a)$$

the mirror symmetry of the velocity field, or the absence of force interaction in the $x = h$ plane:

$$\partial_x v = 0 \quad \text{or} \quad \tau = 0\,; \quad x = h \quad (9b)$$

and delimit the periodic character of the process

$$v(t + 2\pi/\omega, x) = v(t, x)\,. \quad (9c)$$

Brackets in Eq. (8) have been used to denote the odd power-law function:

$$[\xi]^m = |\xi|^m \operatorname{sign}(\xi). \quad (10)$$

This formal agreement shall be adherred to also in the following text.

Normalized kinematic variables shall be introduced so as to avoid appearance of macroscopic parameters in the formulation of the boundary conditions for the problem:

$$X = x/h\,, \quad T = \omega t\,, \quad V = v/(a\omega)\,. \quad (11a,b,c)$$

Substitution of these into Eqs (6), (8), (9a,b,c) leads to the model from the point of view of similarity invariant, in the form

$$\mathrm{Re}\,(\partial_T V - \mathrm{Fr}^{-1}) = \partial_X S \quad (12)$$

$$S = [\partial_X V]^n \quad (13)$$

$$V(T, X)\,|_{X=0} = \cos(T) \quad (14a)$$

$$S(T, X)\,|_{X=1} = 0 \quad (14b)$$

$$(T + 2\pi, X) = V(T, X)\,. \quad (14c)$$

The mathematical model (12)–(14) contains, apart from the index of the flow, $n$, two more independently variable criteria of rheodynamic similarity

$$\mathrm{Re} = \varrho h^{1+n} a^{1-n} \omega^{2-n} K^{-1} = \varrho h a \omega^2 \Big/ \left\{ K \left( \frac{a\omega}{h} \right)^n \right\} \qquad (15a)$$

$$\mathrm{Fr} = a\omega^2 g_z^{-1} = \varrho h a \omega^2 / \varrho g_z h \qquad (16a)$$

expressing characteristic conditions of the inertial forces, $\varrho h a \omega^2$ and oscillatory components of the viscous forces, $K(a\omega/h)^n$ and the steady driving forces of the process $\varrho g_z h$.

For purely pulsating flows ($a' = 0$, $P_0' = P_0$, $g_z' = 0$, $P_s' = P_s$) these criteria may be expressed from Eqs (5), (7) alternatively as

$$\mathrm{Re} = \varrho^n h^{1+n} P_0^{1-n} \omega^n K^{-1} \qquad (15b)$$

$$\mathrm{Fr} = P_0 / P_s \,. \qquad (16b)$$

The exact solution of the mathematical model (12)–(14) has been known only for limiting values of the index of the flow, namely $n = 1$, when the mathematical model remains linear[31], and $n = 0$ when the model reduces to the unidimensional[22] one. For certain combinations of the parameters Re, Fr, asymptotic solutions have been known of a similar problem on PLF in a tube[10,12], which can be easily modified for the given special case of the flow with plane symmetry. In the intermediate region of mean values of Fr, Re results have been known[20,21] of the numerical solution by the finite difference method for discrete values of the parameters Re, Fr.

Our aim has been to study now the dynamics of the flow considered above in the full range of parameters Re, Fr with special attention to all asymptotic regions in the phase plane Re, Fr.

It turns out that physically more instructive and from the standpoint of formulation of asymptotic approximations more efficient appears to regard as the primary result of the solution the field of oscillatory stresses instead of the more commonly used field of velocities. This alternative approach shall be made more clear in the following paragraph.

### *Mechanism of Liquidization, Levels of Stress*

A kinematic prototype of all PLF's is the simple periodic shear flow

$$v_z(t, x) = x\,\gamma(t)\,, \quad \gamma(t + 2\pi/\omega) = \gamma(t) \qquad (17a,b)$$

where the instantaneous shear rate is independent of position. For a given periodically variable shear stress, $\tau(T) = \tau(T + 2\pi)$, $T = \omega t$, of a steady component $\tau_s$ and amplitude of the oscillatory component $\tau_0$

$$\tau(T) = \tau_s + \tau_v(T)\,, \quad \tau_v(T) = \tau_0\,\phi(T)\,, \tag{18a,b}$$

where the normalized characteristic $\phi$ satisfies the relations

$$\bar{\phi} = 0\,, \quad |\phi| \leqq 1\,, \quad \phi(T + 2\pi) = \phi(T) \tag{19a,b,c}$$

the time-averaged shear rate, $\bar{\gamma}$, may be expressed for power-law liquids in the form

$$\bar{\gamma} = (\tau_s/K)^{1/n}\, \mathscr{M}_T([1 + C\,\phi(T)]^{1/n})\,, \tag{20}$$

where

$$C = \tau_0/\tau_s\,.$$

The overbar ($^-$) or the operator ($\mathscr{M}_T$) of the time-averaging operation shall designate in the following equally as in Eqs (*19b*), (*20*) the time-averaged periodic functions with the period $2\pi$:

$$\bar{\xi} = \mathscr{M}_T(\xi(T)) = (2\pi)^{-1} \int_0^{2\pi} \xi(T)\,\mathrm{d}T\,. \tag{21}$$

As mechanically liquidizable generally appear all materials for which, after superimposing an oscillatory component $\tau_v(T)$ on a given steady state component $\tau_s$, a corresponding increase of time-averaged shearing rate $\bar{\gamma}$ appears. Clearly, for an arbitrary $\phi(T)$ with the properties (*19a,b,c*) and an arbitrary $1/n > 1$, $C \neq 0$ the following inequality holds

$$\mathscr{M}_T([1 + C\,\phi(T)]^{1/n}) > 1\,. \tag{22}$$

All pseudoplastic materials are thus liquidizable, and the more so the lower the flow index.

Provided $\phi(T + \pi) = -\phi(T)$ one can find asymptotic representations of the functional (*20*) in the form

$$\mathscr{M}_T([1 + C\,\phi(T)]^{1/n}) \approx \begin{cases} 1 + \alpha_s C^2\,; & C \to 0 \\ \alpha_0 C^{1/n-1}\,; & C \to \infty,\ 1/n \gg 1 \end{cases}, \tag{23a,b}$$

where

$$\alpha_s = \frac{1}{2n}\left(\frac{1}{n} - 1\right)\overline{|\phi(T)|^2} \tag{24a}$$

or

$$\alpha_0 = \frac{1}{n}\overline{|\phi(T)|^{1/n-1}}\,. \tag{24b}$$

As a special case for

$$\phi(T) = [\cos(T)]^m \tag{25}$$

we have

$$\alpha_s = \frac{1}{2n}\left(\frac{1}{n} - 1\right)\frac{\Gamma(m + 1/2)}{\Gamma(m + 1)}\,\pi^{-1/2}\,, \tag{26a}$$

$$\alpha_0 = \frac{1}{n}\,\frac{\Gamma\left(\frac{m}{2}\left(\frac{1}{n} - 1\right) + \frac{1}{2}\right)}{\Gamma\left(\frac{m}{2}\left(\frac{1}{n} - 1\right) + 1\right)}\,\pi^{-1/2}\,. \tag{26b}$$

The course of the function $\beta(n, C) \equiv \mathcal{M}_T([1 + C\cos(T)]^{1/n})$ together with the asymptotic representations in Eqs (*23a,b*), is shown in Fig. 1.

The time-averaged mean velocity of the flow for a plane PLF may be expressed by

$$v_m = h^{-1}\int_0^h \bar{v}\,dx = h^{-1}\int_0^h \bar{\gamma}(x)\,(h - x)\,dx\,, \tag{27}$$

where, similarly as for the simple periodic shearing flow, we have

$$\bar{\gamma}(x) = K^{-1/n}\,\mathcal{M}_T([\tau_s(x) + \tau_0(x)\,\phi(T, x)]^{1/n} \tag{28}$$

and the unsteady component of the stress may be expressed according to Eqs (*6*), (*8*), (*9b*) and (*9c*) explicitly in the form

$$\tau_s(x) = \varrho\, g_z(h - x)\,. \tag{29}$$

As a special case for a steady flow without the oscillatory component we have

$$v_s = h(\varrho g_z\, h/K)^{1/n}/(1/n + 2)\,. \tag{30}$$

A practical quantitative criterion of the extent of liquidization under the PLF is the degree of liquidization $I$ defined as the ratio of the time-averaged mean velocity $v_m$ under the given conditions and comparable conditions without vibrations.

For a plane PLF of a power-law liquid the degree of liquidization, $I$, may, thus be expressed from Eqs (*27*), (*28*), (*29*) and (*30*) as

$$I = (1/n + 2)\int_0^1 \mathcal{M}_T([(1 - X) + \text{Fr}\,\sigma(T, X)]^{1/n})\,(1 - X)\,\mathrm{d}X\,, \tag{31}$$

where the normalized oscillatory component of stress has been denoted by $\sigma(T, X)$

$$\sigma = \tau_v/\varrho a\omega^2 h = \text{Re}^{-1}([\partial_X V]^n - \mathcal{M}_T([\partial_X V]^n))\,. \tag{32}$$

It turns out that not only for the velocity field $V(T, X)$ but also for the field of oscillatory stresses $\sigma(T, X)$ one can formulate a boundary value problem of parabolic type. From Eqs (*13*) and (*14a*) the velocity field may be expressed in terms of stresses as

$$V(T, X) = \cos(T) + \int_0^X [S]^{1/n}\,\mathrm{d}X\,, \tag{33}$$

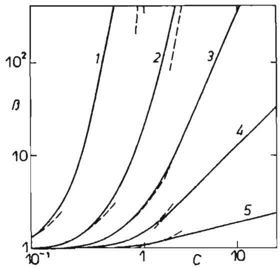


FIG. 1

Function $\beta(n, C)$

*1* $n = 0{\cdot}05$, *2* $n = 0{\cdot}15$, *3* $n = 0{\cdot}30$, *4* $n = 0{\cdot}5$, *5* $n = 0{\cdot}8$. Solid lines represent exact courses, broken lines the asymptotes (*26a,b*).

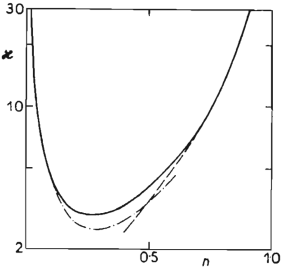


FIG. 2

Function $\varkappa(n)$

Solid lines represent function in Eq. (*75*), broken lines asymptotes $\varkappa \approx (2 + 1/(2n)) \,.\, (2{\cdot}16n)^{3n}$, or $\varkappa \approx 3/(1 - n) - 2{\cdot}58$, for $n \to 0$ (dash-and-dot-line) or for $n \to 1$ (broken line).

where, according to Eqs (*13*), (*29*) and (*32*) we have

$$S = \mathrm{Re}\,\mathrm{Fr}^{-1}(1 - X) + \mathrm{Re}\,\sigma\,. \tag{34}$$

Substitution of these expressions into Eq. (*12*) and some rearrangement leads to the following parabolic equations

$$\mathrm{Re}^{1/n}\,\partial_T[\mathrm{Fr}^{-1}(1 - X) + \sigma]^{1/n} = \partial^2_{XX}\sigma\,. \tag{35}$$

The boundary conditions for $\sigma$ may be formulated in the form

$$\partial_X\sigma = -\sin(T)\,; \quad X = 0 \tag{36a}$$

$$\sigma = 0\,; \qquad X = 1 \tag{36b}$$

$$\sigma(T + 2\pi, X) = \sigma(T, X)\,. \tag{36c}$$

For a known $\sigma(T, X)$ the kinematic quantities may be determined from Eqs (*13*), (*31*), (*33*), *etc.*

*Rheodynamic Regimes*

Compared to the common formulation of the rheodynamic models in terms of the velocity field the formulation in terms of oscillatory stress exhibits certain advantages.

From the analytical standpoint it is interesting that the model of the flow in stresses has no singularities at the point $S = 0$ while for $0 < n < 1$ corresponding velocity-based model these singularities clearly possesses. This is of consequence in the accuracy of integration by the finite difference method, utilizing essentially local representations of corresponding fields by Taylor series. As shall be seen, another advantage of the representation based on the equation of motion in stresses is the option of explicit formulation of asymptotic approximations.

From physical point of view the model in stresses is more instructive as it immediately indicates physical meaning of individual rheodynamic criteria of similarity in connection with the existence of a few qualitatively different rheodynamic regimes:

*The creeping flow regime*, $\mathrm{Re} \to 0$ the field $\sigma(T, X)$ clearly has the asymptotic structures as follows

$$\sigma = (1 - X)\sin(T)\,. \tag{37}$$

The ratio of the oscillatory and the steady state stress, $C$, in this case is given explicitly by

$$\tau_0/\tau_s = \mathrm{Fr}\,. \tag{38}$$

*Boundary layer regime,* Re $\to \infty$. Typical feature is a rapid decrease of the oscillatory stress with increasing distance from the wall, $\partial^2_{XX}\sigma|_{X=0} = O(\mathrm{Re}^{1/n})$ and independence of the phenomena in the proximity of the wall on the overall thickness of the liquid layer, $h$. The structure of the model of the flow in region Re $\to \infty$ shall be studied in detail in the following.

Apart from these classic regimes it appears useful to distinguish between two more asymptotic regimes according to the ratio of the steady and the oscillatory stress component. From asymptotic relations (*23a,b*) for the simple periodic shear flow it is apparent that if $\tau_0/\tau_s = C \ll 1$ the liquidization would remain only a secondary effect while in the opposite case, $C \gg 1$ the liquidization may become dramatic even in region of high values of Re. These two different situations have in the general case of PLF two characteristic asymptotic regimes with correspondingly simplified asymptotic models of the flows in terms of stress.

The region of parameters Re, Fr for which Fr $|\sigma| \ll 1$ shall be designated as the regime of linear dynamics of oscillations. The non-linear parabolic equations (*35*) may be in this case linearized to the form

$$n^{-1}(\mathrm{Re}\,\mathrm{Fr}^{-1+n})^{1/n}(1 - X)^{1/n-1}\,\partial_T\sigma = \partial^2_{XX}\sigma \tag{39}$$

and an explicit closed form solution[13] may be obtained.

In region of the parameters Re, Fr, where, on the contrary Fr $\gg 1$, one may neglect the effect of the steady-state component of the stress on the structure of the field of oscillatory stress and Eq. (*35*) may be simplified to a form typical for the regime of pure oscillations

$$\mathrm{Re}^{1/n}\,\partial_T[\sigma]^{1/n} = \partial^2_{XX}\sigma\,. \tag{40}$$

In the following we shall examine in detail the problem of four regimes, and localize corresponding regions in the phase plane (Re, Fr).

### *Region of Low* Re

For the determination of the analytical approximation of the solution in region of low Re one can utilize the method of iterative integration in the form

$$\sigma_{i+1} = (1 - X)\sin(T) - \mathrm{Re}^{1/n}\int_X^1 \mathrm{d}X_1 \int_0^{X_1} \partial_T[\mathrm{Fr}^{-1}(1 - \xi) + \sigma_i(T, \xi)]^{1/n}\,\mathrm{d}\xi \tag{41}$$

where, for instance, $\sigma_0 = 0$. For Re $\rightarrow$ 0 this recurrent formula leads to a functional expansion with respect to integer powers of the parameter $(\mathrm{Re}^{1/n})$.
As a special case

$$\sigma_1 = (1 - X)\sin(T) - \mathrm{Re}^{1/n}\, 1/n \cos(T)\left|\mathrm{Fr}^{-1} + \sin T\right|^{1/n-1} \mathrm{F}_n(X)\,, \qquad (42)$$

where

$$\mathrm{F}_n(X) = \int_X^1 \mathrm{d}X_1 \int_0^{X_1} (1 - \xi)^{1/n}\, \mathrm{d}\xi\,. \qquad (43)$$

For Re $\ll$ 1 one thus can express the degree of liquidization in a simple form as

$$I \approx \beta(n, \mathrm{Fr}) + \mathrm{O}(\mathrm{Re}^{1/n})\,. \qquad (44)$$

*The Region of Linear Oscillations*

In region where $|\sigma|$ Fr $\ll$ 1, one can linearize with respect to $\sigma$ the expression $[(\mathrm{Fr}^{-1}(1 - X) + \sigma]^{1/n}$ and arrive at the linear model for $\sigma(T, X)$, see Eq. (*39*), (*36a,b,c*). Substitution

$$\sigma(T, X) = \sin(T) f_s(X) + \cos(T) f_c(X) \qquad (45)$$

reduces this linear model to an ordinary boundary value problem

$$n(1 - X)^{1/n-1} f_s - M\, \mathrm{d}^2_{XX} f_c = 0 \qquad (46a)$$

$$n(1 - X)^{1/n-1} f_c + M\, \mathrm{d}^2_{XX} f_s = 0 \qquad (46b)$$

$$M = (\mathrm{Fr}^{1-n}\, \mathrm{Re}^{-1})^{1/n} \qquad (47)$$

with the boundary conditions

$$f_s(1) = 0\,, \quad f_c(1) = 0\,, \quad \mathrm{d}_X f_c(0) = 0\,, \quad \mathrm{d}_X f_s(1) = -1\,. \qquad (48a,b,c,d)$$

The linearized formulation of the problem according to Eqs (*46*), (*47*) contains, for a given value of the flow index $n$, a single variable parameter $M$. Let an *a priori* requirement regarding the accuracy of the linearization be given by the inequality

$$f(M) = \mathrm{Max}_{(T,X)} \left|\sigma_0(T, X)\right| \leqq \mathrm{Fr}^{-1}\, \varepsilon(n)\,, \qquad (49a)$$

where $\sigma_0(T, X)$ is the solution of the linear problem (*46*), (*47*). The expression on the left hand side of Eq. (*49*) is clearly dependent on the parameter $M$ only, and, consequently, one can always find for given $M$ = const. such a small Fr for which the inequality (*49*) rewritten into the form

$$\mathrm{Fr} \leqq \varepsilon(n)/f(M) \qquad (49b)$$

is fulfilled.

For Re → 0 clearly the solution according to (*46*), (*48*) leads to the already known result (*37*) and the condition (*49*) of linearizability takes in this region the following form

$$\mathrm{Fr} \leqq \varepsilon(n)\,, \quad \mathrm{Re} \to 0\,. \qquad (50)$$

For Re → ∞ clearly the solution for $X \to 0$ takes an exponential character

$$\sigma \approx c^{-1} \exp\left(-cX\right) \sin\left(T - cX - \pi/4\right), \qquad (51)$$

where

$$c = (M/2)^{1/2}\,. \qquad (52)$$

The condition of linearizability thus may be expressed in this region from Eqs (*49*) and (*52*) explicitly in the form

$$\mathrm{Fr} \leqq (2\varepsilon^2/n)^{n/(1+n)}\,\mathrm{Re}^{1/(1+n)}\,. \qquad (53)$$

The course of the critical curve $\mathrm{Fr} = \mathrm{Fr}_\varepsilon(\mathrm{Re})$ in region of intermediate values of Re may be found only by solving the linear problem in the whole region of parameter $M$ and by solving the set of Eqs (*47*), (*49*) just like carried out by Gianetto and Baldi[10] for the case of pulsating flow in a tube. These authors, however, did not formulate corresponding asymptotic relations of the type (*50*) and (*53*) and therefore overlooked the existence of the linear region for Re → 0, $\mathrm{Fr} < \varepsilon$.

From the standpoint of liquidization the regimes in region of the linear dynamics of oscillations lack importance, because, according to the praemisae the terms of the order $I - 1 = 0(\mathrm{Fr}^2\,\sigma^2)$ are negligible for the required accuracy $\mathrm{Fr}^2\,\sigma^2 = 0(\varepsilon^2)$.

### *Quasi-Oscillatory Regimes*

In case of $\mathrm{Fr}^{-1} = 0$ the equation of motion (*35*) reduces identically to the form (*40*), corresponding to the case of pure oscillations with zero steady-state stress component. The problem simplifies not only in that there remains only a single variable parameter, Re, but also in that the field of $\sigma(T, X)$ takes two more symmetries with respect to the time variable. The first of these

$$\sigma(T + \pi, X) = -\sigma(T, X) \qquad (54)$$

halves the necessary integration region of the time variable for $T \in (0; 2\pi)$ to $T \in (0; \pi)$. The second has the character of the mirror symmetry

$$\sigma(T_0(X) + T, X) = \sigma(T_0(X) - T, X), \quad (55)$$

where, of course, one has to solve the complete problem in order to determine $T_0(X)$. This symmetry is useful as a check of, for instance, numerical solutions by the finite difference method.

For $\mathrm{Fr}^{-1} \neq 0$ one can speak of quasi-oscillatory regime for those regions of (Re, Fr) where the true course of the field $\sigma(T, X)$ does not deviate appreciably from that for $\mathrm{Fr}^{-1} = 0$, *i.e.* where existence of the non-zero steady-state component would not influence significantly the field of oscillatory stress.

From Eq. (*35*) it is apparent that one of these quasi-oscillatory regions is the region of very low values of Re. As a special case, in the neighbourhood of Re = 0 the regular asymptotic representation (*42*) holds, according to which the steady state component is represented by the term of the order $0(\mathrm{Re}^{1/n}(1 + \mathrm{Fr}^{-1})^{1/n-1})$. For finite Fr thus the effect of the steady-state component is of the order $0(\mathrm{Re}^{1/n})$ and for $\mathrm{Fr} \to 0$ of the order $(\mathrm{Re}/\mathrm{Fr})^{1/n}\,\mathrm{Fr}$. Excepting the last region, which is identical with the region of the linear dynamics of oscillations, then for $\mathrm{Re} \ll 1$ the whole region $\mathrm{Fr} > 1$ is the region of quasi-oscillatory regime.

In region of intermediate and higher values of Re the purely oscillatory approximation may be justified in region where for the selected accuracy $\varepsilon(n)$ one may write

$$|\partial_T[\mathrm{Fr}^{-1}(1 - X) + \sigma]^{1/n} - \partial_T[\sigma]^{1/n}| \leqq \varepsilon(n)|\,\partial^2_{XX}\sigma|\,. \quad (56)$$

On approximating the expression on the LHS by the first term of the binomial expansion we arrive at

$$(1 - X)\,|\partial_T\sigma_0^{1/n-1}| \leqq \mathrm{Fr}\,\varepsilon(n)|\,\partial^2_{XX}\sigma_0|\,, \quad (57)$$

where $\sigma_0$ is for the given Re a solution of the purely oscillatory problem, $\mathrm{Fr}^{-1} = 0$. From the inequality (*57*) it is apparent that for each Re there exists such an Fr that the oscillatory approximation leads to a satisfactory estimate of the field $\sigma \approx \sigma_0$.

In region where $C = \mathrm{Fr}(1 - X)\,\mathrm{Max}\,|\sigma| \gg 1$, one can estimate $I$ from the asymptotic representation (*23b*) with the result[31]

$$I \approx (1/n + 2)\,\mathrm{Fr}^{1/n-1}\int_0^1 (1 - X)^{1/n}\,\mathscr{M}_T(|\sigma|^{1/n-1})\,\mathrm{d}X\,, \quad (58)$$

where $\sigma = \sigma(T, X; \mathrm{Re}, n)$.

*Boundary Layer Regimes*

High values of Re may be achieved either by increasing the overall thickness of the material, $h$, or, by increasing the intensity of vibrations, characterized in the definition of Re by the complex $(a^{1-n}\omega^{2-n})$ or $(P_0^{1-n}\omega^n)$. If at a constant steady state stress on the oscillating wall, $\tau_s = \varrho g_z h$, $h$ grows to infinity, the concept of the process near the wall as being independent of the thickness $h$ becomes physically acceptable. The results for $h \to \infty$, *i.e.* Re $\to \infty$ for a given Fr, however, must have their corresponding physical interpretation of the normalized model even for the case when the growth of Re is achieved through the changes of intensity of oscillations. A unified approach offers again the theory of similarity. Instead of $h$ one can choose another length scale of the problem, $L_B$

$$L_B = \left(K/(\varrho a^{1-n}\omega^{2-n})\right)^{1/(1+n)} = h \, \mathrm{Re}^{-1/(1+n)} \tag{59}$$

for the normalized description of the relevant physical field in terms of the normalized geometrical variable

$$Y = x/L_B = \mathrm{Re}^{1/(1+n)} X . \tag{60}$$

Substitution of this new independent variable and the new normalized oscillatory stress

$$\vartheta = \tau/\tau_B = \mathrm{Re}^{1/(1+n)} \sigma , \tag{61}$$

where

$$\tau_B = \left(K(\varrho a^2 \omega^3)^n\right)^{1/(1+n)} \tag{62}$$

into the original model (*34*), (*36*) leads to a new set for the field $\vartheta(T, Y)$ as

$$\partial_T[B^{-1}(1 - Y/H) + \vartheta]^{1/n} = \partial^2_{YY}\vartheta \tag{63}$$

$$\partial_Y \vartheta = -\sin T ; \; Y = 0 \tag{64a}$$

$$\vartheta = 0 ; \quad Y = \mathrm{H} \tag{64b}$$

$$\vartheta(T + 2\pi, Y) = \vartheta(T, Y) , \tag{64c}$$

where

$$H = h/L_B = \mathrm{Re}^{1/(1+n)} , \tag{65}$$

$$B = \tau_B/(\varrho g_z h) = \mathrm{Fr}/H . \tag{66}$$

This new normalization shows that for $H \to \infty$, *i.e.* for $\mathrm{Re} \to \infty$, there exists an asymptotic solution, $\vartheta_0$, of the problem (*63*), (*64*) defined by the boundary value problem

$$\partial_T [B^{-1} + \vartheta_0]^{1/n} = \partial^2_{YY} \vartheta_0 \qquad (67)$$

$$\partial_Y \vartheta_0 = -\sin T ; \quad Y = 0 \qquad (68a)$$

$$\vartheta_0 \to 0 ; \quad Y \to \infty \qquad (68b)$$

$$\vartheta_0(T + 2\pi, Y) = \vartheta_0(T, Y) , \qquad (68c)$$

where the ratio of the steady-state and oscillatory stresses is characterized by the parameter $B$. This asymptotic solution possesses a reasonable physical meaning for the region in the neighbourhood of the oscillating wall, *i.e.* for finite values of $Y$ for $\mathrm{Re} \to \infty$.

In region $\mathrm{Re} \gg 1$ the usability of the purely oscillatory approximation is clearly constrained by the condition of the type

$$B^{-1} \leqq \varepsilon_1(n) \qquad (69a)$$

and the usability of the linear approximation by the constraint of the type

$$B^{-1} \gtreqless \varepsilon_2^{-1}(n) \qquad (69b)$$

which is a condition which has been arrived at *via* different route in the part regarding linear approximations, see Eq. (*54*).

The regimes in region $\mathrm{Re} \to \infty$ are looked upon as boundary layer ones owing to the fact that the amplitude of the oscillatory stress rapidly decays with increasing distance from the oscillating wall. As a special case for $n = 1$, $\mathrm{Re} \to \infty$, the exact solution of the problem (*67*) has been known in the form

$$\vartheta_0(T, Y) = -\exp(-cY)\cos(T - cY + \pi/4) , \quad c = 2^{-1/2} . \qquad (70)$$

For $0 < n \leqq 1$ the amplitude of the stress decreases again roughly exponentially[10,31] and may be neglected outside the region of the boundary layer of thickness

$$\delta = \lambda(n)\, L_B \qquad (71)$$

while approximately we can write[31-34] $\lambda \approx 4$.

From the standpoint of the structure of the velocity field for PLF in region of higher Re one can distinguish the region of the oscillatory boundary layer, the one being under the effect of the liquidazing oscillations, and the outer region of the steady flow without any appreciable effect of the oscillations.

From the standpoint of liquidization the interesting region for Re $\gg 1$ is only the region of the quasi-stationary boundary layer, $B^{-1}$, when, according to Eqs (*60*), (*61*), (*67*) and (*70*) one may write

$$\sigma(T, X) \approx \begin{cases} H^{-1}\,\vartheta_0(T, HX)\,; & HX < \lambda \\ 0\,; & HX > \lambda\,. \end{cases} \tag{72}$$

The degree of liquidization $I$, for Re $\rightarrow \infty$, $B \gg 1$, $1/n \gg 1$ may be approximated according to Eqs (*72*) and (*24b*) as follows

$$I = \left(\frac{1}{n} + 2\right) \lim_{\substack{H \to \infty \\ B \gg 1}} \int_0^1 \mathcal{M}_T([(1 - X) + B\,\vartheta_0(T, HX)]^{1/n})\,(1 - X)\,\mathrm{d}X \approx$$

$$\approx 1 + \lim_{H \to \infty} \int_0^{\lambda/H} (1 + 2n)/n^2 B^{1/n-1}(1 - X)^{1/n-1}\,\mathcal{M}_T(|\vartheta_0(T, HX)|^{1/n-1})\,\mathrm{d}X =$$

$$= 1 + H^{-1}B^{1/n-1}\,\varkappa(n) = 1 + \varkappa(n)\,(\mathrm{Fr}\,\mathrm{Re}^{-1/(1-n^2)})^{1/n-1}\,, \tag{73}$$

where

$$\varkappa(n) = (1 + 2n)/n^2 \int_0^{\lambda} \mathcal{M}_T(|\vartheta_0(T, Y)|^{1/n-1})\,\mathrm{d}Y\,. \tag{74}$$

The approximate values $\varkappa(n)$ according to the results of the work[35] may be expressed for $1/n \gg 1$ by the function

$$\varkappa(n) \approx \frac{(1 + 2n)(1 + n)}{2n(1 - n)} \left(\frac{(2\Gamma(1/2n)}{(1 + n)\sqrt{\pi}\,\Gamma(1/2n + 3/2)}\right)^{2n/(1+n)} \tag{75}$$

whose course has been shown in semi-log coordinates in Fig. 2.

## DISCUSSION AND CONCLUSION

The rheodynamic regime for PLF of power-law liquids for a given flow index $n$ has been determined by the values of two independent rheodynamic criteria Re, Fr, *i.e.* by the position of the point (Re, Fr) in the phase plane shown in Fig. 3.

From the view point of the analytical approximations it is useful to distinguish four typical regions in the phase plane (Re, Fr) marked in Fig. 3 by letters *C*, *B*, *L*, *O*:

The region of the creeping flow $C$ for $\mathrm{Re} \ll 1$, the boundary layer flow region $B$ for $\mathrm{Re} \gg 1$, the region of quasi-linear dynamics L for $C \ll 1$ and the region of quasi--oscillatory dynamics $O$ for $C \gg 1$. The parameter $C$, indicating the ratio of the amplitude of the oscillatory and the steady stress in the proximity of the wall may be expressed asymptotically by the following relations

$$C \approx \begin{cases} \mathrm{Fr} & \text{for} \quad \mathrm{Re} \ll 1 \\ \mathrm{Fr}\,\mathrm{Re}^{-1/(1+n)} & \text{for} \quad \mathrm{Re} \gg 1\,. \end{cases} \tag{76}$$

Each of the given regions has its own corresponding asymptotic model of the flow with a single independently variable criterion of rheodynamic similarity. Summarily these data regarding the regimes $C$, $L$, $B$, $O$ are given in Table I.

In the corners of the phase diagram (Re, Fr) there are regions corresponding to four different asymptotic rheodynamic regimes:

The creeping flow linear oscillation regime ($CL$) for $\mathrm{Fr} \ll 1$, $\mathrm{Re} \ll 1$, the boundary layer linear oscillation regime ($BL$) for $\mathrm{Fr}\,\mathrm{Re}^{-1/(1+n)} \gg 1$, $\mathrm{Re} \gg 1$, the boundary layer pure oscillation regime ($BO$) for $\mathrm{Fr}\,\mathrm{Re}^{-1/(1+n)} \gg 1$, $\mathrm{Re} \gg 1$ and the creeping flow pure oscillation regime ($CO$) for $\mathrm{Fr} \gg 1$, $\mathrm{Re} \ll 1$. Principal data on these regimes are summarized in Table II.

The degree of liquidization, $I$, for pseudoplastic liquids $n < 1$, may take essentially arbitrarily high values. A maximum liquidization for a given Fr can be achieved for $\mathrm{Re} \to 0$ when

$$\lim_{\mathrm{Re} \to 0} I(\mathrm{Re}, \mathrm{Fr}) = \beta(n, \mathrm{Fr})\,. \tag{77}$$

TABLE I

Basic Rheodynamic Regimes under PLF of Purely Viscous Materials

| Regime | Condition of existence | Independent variable | Degree of liquidization | Oscillatory wall shear stress | Depth of penetration of oscillations from the wall |
|---|---|---|---|---|---|
| $C$ | $\mathrm{Re} \ll 1$ | Fr | $f_c(\mathrm{Fr})$ | $\tau_c$ | $h$ |
| $L$ | $\mathrm{Fr} \ll 1$ or $B \ll 1$ | $M$ | 1 | — | — |
| $B$ | $\mathrm{Re} \gg 1$ | $B$ | $1 + \mathrm{Re}^{-1/(1+n)} f_B(\mathrm{B})$ | $\tau_B$ | $L_B$ |
| $O$ | $\mathrm{Fr} \gg 1$ and $B \gg 1$ | Re | $\mathrm{Fr}^{1/n-1} f_0(\mathrm{Re})$ | — | — |

The course of this function has been shown in Fig. 1. Also for the (*BO*) regime one can find a simple explicit expression for $I$ in the form

$$\lim_{\substack{\mathrm{Re}\to\infty \\ \mathrm{B}\to\infty}} I\,(\mathrm{Re}, \mathrm{Fr}) = 1 + \varkappa(n)\,(\mathrm{Fr}^{1-n}\ \mathrm{Re}^{-1/(1+n)})^{1/n}\,, \tag{78}$$

where the function $\varkappa(n)$, approximately determined in Eq. (75) is shown in Fig. 2.

Based on these asymptotic approximations the course of $I(\mathrm{Re}, \mathrm{Fr})$ for $n = 1/3$ is shown qualitatively in Fig. 4. In region of linear oscillations the quantity $(I - 1)$ is very small, in the order of magnitude identical with the error of the linear approximation. A more precise data on the degree of liquidization $I(\mathrm{Re}, \mathrm{Fr}, n)$ require

TABLE II

Asymptotic Rheodynamic Regimes under PLF of Purely Viscous Materials

| Regime | Condition of existence | Wall stress characteristic | Degree of liquidization |
|---|---|---|---|
| *CL* | $\mathrm{Fr} \ll 1$<br>$\mathrm{Re} \ll 1$ | $\tau_s$ | $1 + 0(\mathrm{Fr}^2)$ |
| *BL* | $\mathrm{B} \ll 1$<br>$\mathrm{Re} \gg 1$ | $\tau_s$ | $1 + 0(B^2)$ |
| *CO* | $\mathrm{Fr} \gg 1$<br>$\mathrm{Re} \ll 1$ | $\tau_c$ | $1 + \mathrm{Fr}^{1/n-1}\ \mathscr{M}_T(\lvert\sin T\rvert^{1/n-1})$ |
| *BO* | $\mathrm{B} \gg 1$<br>$\mathrm{Re} \gg 1$ | $\tau_B$ | $1 + \varkappa(n)\ \mathrm{Fr}^{1/n-1}\ \mathrm{Re}^{1/(n+n^2)}$ |

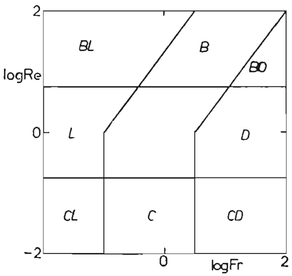


FIG. 3

Rheodynamic Regimes under Periodic Longitudinal Flows in the Phase Plane (Re, Fr) numerical solution of the relevant mathematical models and have not been made available so far in the full range of parameters. For $n = 1/3$ Table III summarizes some numerical and experimental results[20] on plane PLF's. The experiments have been carried out with the gravitational flow of suspensions in glycerol solution

TABLE III

Degree of Liquidization under the Film Flow of Kaoline Suspension in Water Solution of Glycerol[20], $n = 0{\cdot}33$, $\varrho = 1420$ kg m$^{-3}$, down a Vertical Oscillating Plane Wall, $g_z = 9{\cdot}81$ m s$^{-2}$

$K/\varrho$ kinematic coefficient of consistency [m$^2$ s$^{n-2}$], $Q$ intensity of irrigation [m$^2$ s$^{-1}$], $h_0$ film thickness under flow without vibrations [m], $\omega$ angular frequency of the oscillations [rad s$^{-1}$], $a$ amplitude of the oscillations [m].

| $10^4 K/\varrho$ m$^2$ s$^{n-2}$ | $10^6 Q$ m$^2$ s$^{-1}$ | $10^4 h_0$ m | $\omega$ rad s$^{-1}$ | $10^5 a$ m | Re | Fr | $I_{exp}$ [a] | $I_{num}$ [b] | $I_{max}$ [c] |
|---|---|---|---|---|---|---|---|---|---|
| 143 | 31 | 34 | 364 | 8 | 1·01 | 1·03 | 2·0 | 1·8 | 2·6 |
| | | | | 19 | 1·38 | 2·56 | 6·2 | 6·4 | 10·8 |
| | | | | 176 | 1·90 | 7·49 | 27·6 | 36·1 | 85·2 |
| 138 | 49 | 37 | 369 | 7 | 1·23 | 1·04 | 1·6 | 1·6 | 2·6 |
| | | | | 38 | 2·16 | 5·26 | 9·4 | 15·2 | 42·5 |
| 137 | 48 | 36 | 251 | 11 | 0·86 | 0·71 | 1·4 | 1·3 | 1·8 |
| | | | | 22 | 1·17 | 1·42 | 2·6 | 2·5 | 4·0 |
| | | | | 56 | 1·59 | 3·61 | 8·6 | 10·5 | 20·6 |
| | | | | 92 | 1·83 | 5·94 | 17·8 | 23·9 | 53·9 |
| 132 | 68 | 38 | 251 | 21 | 1·35 | 1·38 | 2·0 | 2·1 | 3·9 |
| | | | | 56 | 1·84 | 3·61 | 5·9 | 8·6 | 20·6 |
| | | | | 99 | 2·19 | 6·37 | 12·9 | 21·6 | 61·9 |
| 133 | 68 | 38 | 440 | 12 | 2·20 | 2·41 | 2·6 | 3·1 | 9·7 |
| | | | | 44 | 3·68 | 8·67 | 9·5 | 17·5 | 114 |
| | | | | 81 | 4·62 | 16·0 | 19·2 | 40·5 | 385 |
| 132 | 19 | 29 | 421 | 4 | 0·77 | 0·72 | 1·7 | 1·4 | 1·8 |
| | | | | 43 | 1·65 | 7·70 | 39·7 | 17·4 | 90·0 |
| 141 | 9 | 26 | 433 | 4 | 0·62 | 0·76 | 1·9 | 1·6 | 1·9 |
| | | | | 19 | 0·89 | 3·44 | 23·0 | 16·3 | 18·8 |
| | | | | 39 | 1·20 | 7·11 | 63·5 | 57·3 | 76·8 |
| 142 | 9 | 26 | 207 | 8 | 0·32 | 0·35 | 1·3 | 1·2 | 1·2 |
| | | | | 27 | 0·57 | 1·17 | 3·2 | 2·6 | 3·0 |
| | | | | 80 | 0·65 | 3·51 | 30·7 | 19·2 | 19·5 |
| 120 | 114 | 30 | 377 | 22 | 2·85 | 3·19 | 2·9 | 3·5 | 16·3 |
| | | | | 92 | 4·85 | 13·3 | 12·0 | 25·3 | 266 |

[a] $I_{exp}$ experimental degree of liquidization, [b] $I_{num}$ degree of liquidization computed numerically[20] from the theoretical model Eq. (*12*)—(*15*), [c] $I_{max}$ upper limit of the estimated degree of liquidization for Re $\to \infty$, Eq. (*44*).

on a vertical oscillating sheet. The agreement with the experimental data and the results of the solution of the corresponding parabolic problems in terms of velocities is satisfactory. Because, however, all experiments correspond to the region of intermediate values of Re, Fr, none of the asymptotic representations is suitable for the correlation of the degree of liquidization. Empirically, these data may be correlated similarly as in case[36] $n = 0{\cdot}15$, in dependence on a single criterion (Re/Fr), see Fig. 5.

The possibility of mechanical liquidization of pseudoplastic materials is technically interesting particularly in combination with the gravitational flow, *i.e.* liquidization by the oscillatory motion of the walls of the duct. The walls of the duct can be set into the oscillatory motion either by increased amplitude, which is technically limited to roughly millimeters, or, by increased frequency of the oscillations. The change of each of these parameters becomes manifest through the simultaneous change of the values of Fr and Re. Fig. 6 shows the trends of the variations of Fr, Re in dependence on the variations of the process parameters $P_0$, $a$ or $\omega$. From the figure it is apparent that a gradual increase of $a$ or $\omega$ leads in either case to the region of high values of Re where the liquidization is less effective, usually into the region (BO) in which, according to Eq. (*78*) we may write asymptotically (for given $n$, $K$, $\varrho$, $g_z h$)

$$I - 1 \sim \left(a^{1-n}\omega^{1-2n}\right)^{1/(1+n)} . \qquad (79)$$

From this relationship it is apparent that increasing $\omega$ for $n > 0{\cdot}5$ causes the degree of liquidization in region $\mathrm{Re} \gg 1$ to decrease and hence that for a given $a$ there exists a single optimum frequency of oscillations for which $I$ reaches maximum.

Based on the qualitative theory of mechanical liquidization one can summarize as follows: *a*) A feasible degree of liquidization significantly increasing with decreasing

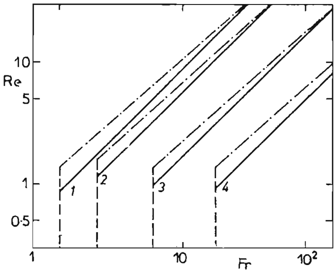


FIG. 4

Degree of Liquidization in the Phase Plane (Re, Fr) for $n = 1/3$

The figure shows the contours of constant liquidization (*1* $I = 1{\cdot}5$, *2* $I = 5{\cdot}0$, *3* $I = 50$, *4* $I = 500$) according to the asymptotic expression (*77*) for $\mathrm{Re} \to 0$ (broken line), according to the asypmtotic expression (*78*) for $\mathrm{Re} \to \infty$, $B \to \infty$ (dash-and-dot line) and according to the empirical correlation $I = 1{\cdot}83\,(\mathrm{Fr}/\mathrm{Re})^{1{\cdot}86}$ obtained by processing experimental data[20] solid line), see also Table III.

flow index as well as with increasing coefficient of consistency which allows a sufficient level of $\tau_0$ to be reached in region of low Re. In region of extremely low flow indices, say for $n < 0{\cdot}5$ a plausible degree of liquidization may be characterized by $I \approx 10$ to $10^6$. *b*) In region of high flow indices, say fori $n > 0{\cdot}8$, mechanical liquidization cannot be expected to reach any technically significant level.

Typical materials in case of which one can successfully utilize the principles of mechanical liquidization are colloid dispersions of biological provenience and fine inorganic suspensions (pigments, clays) of viscoplastic character.

The presented theory of PLF has been confined to rheodynamically stable regimes of purely viscous materials with the power-law representation of the viscosity function. It is thus restricted by numerous assumptions which in real configurations need not be met.

The restriction to rheodynamically stable regimes reflects especially in that these regimes are presumably controlled by only two criteria, the oscillatory Fr and the oscillatory Re. The source of instabilities may be either instabilities of the steady state component of the flow, controlled by the criterion $(v_m^2/g_z h)$ or the instabilities of the oscillatory component of the flow, controlled by the criterion $(\varrho a^2 \omega^{2-n}/K)$, or, instabilities of combined type. These criteria have not been incorporated into the presented analysis and cannot be expressed by any combination of Re and Fr.

The assumption of the power-law representation of the viscosity function in itself is not a basic defect[17,27]. One has to only bear in mind that the parameters $K$, $n$ represent with sufficient fidelity the real course of the viscosity function in only a limited region of parameters. For a concrete utilization of the results of the theoretical model with the power-law representation of the viscosity function it is therefore necessary to determine $K$ and $n$ for the given real liquid on the basis of the viscosity

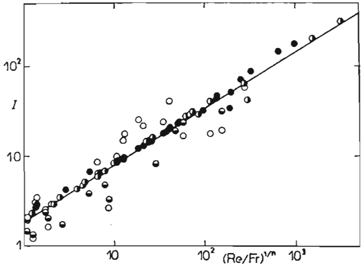


FIG. 5

Empirical Correlation of Data[20] for $n = 0{\cdot}33$

◒ Data from the region Re < 1, ◑ data from 1 < Re < 2, ● data from Re > 2, ○ numerically generated data, see Table III. The straight line represents the equation $I = 1{\cdot}83\,(\mathrm{Re}/\mathrm{Fr})^{1.86}$.

data from the interval of the shear stress corresponding to the maximum stress on the wall for the PLF under consideration. This characteristic stress may be estimated from

$$\tau \approx \begin{cases} \varrho g h + (K(\varrho a^2 \omega^3)^n)^{1/(1+n)}\,; & \mathrm{Re} > 1 \\ \varrho g h + \varrho a \omega^2 h\,; & \mathrm{Re} < 1 \end{cases}. \qquad (80)$$

As a certain defect of the rheodynamic theories of PLF starting from constitutive models of purely viscous type is that they ignore time-dependent effects. Available experimental data on PLF of polymer systems exhibit fairly good agreement with

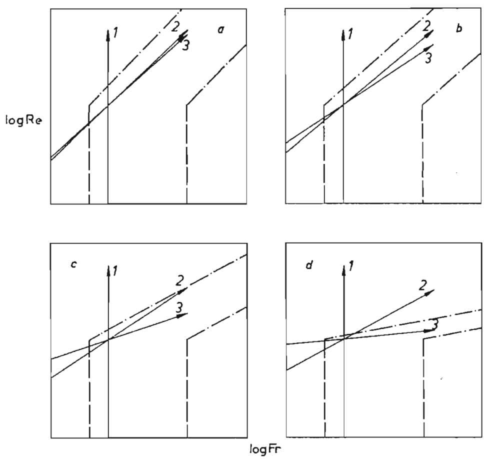


FIG. 6
The Effect of Process Parameters $P_0$, $a$ and $\omega$ on the Degree of Liquidization for *a* $n = 0{\cdot}1$, *b* $n = 1/3$, *c* $n = 2/3$, *d* $n = 0{\cdot}9$

The contours $I =$ const. for the boundary layer regime are shown by the dash-and-dot line, in the creeping flow regime by the broken lines $I_1 < I_2$. Oriented straight lines show the course of Fr and Re for the case of increasing $\omega$ at $P_0 =$ const. (straight line *1*), for the case of increasing $P_0$ at $\omega =$ const. (straight line *3*), for the case of increasing $a$ and $\omega =$ const. (straight line *3*) and for the case of increasing $\omega$ at $a =$ const. (straight line *2*).

purely viscous theories; the viscoelastic effects seem to diminish somewhat the degree of liquidization estimated according to purely viscous theories[11,20]. However, one has to bear in mind that the available data cover a rather narrow section of the phase plane (Re, Fr) in the neighbourhood of the point Re = 1, Fr = 1. Theoretical papers on PLF of viscoelastic materials[1-8] have been devoted so far exclusively to the asymptotic linear region $C \ll 1$ and do not therefore bring a desirable theoretical explanation of the effect of nonlinear viscoelastic effects on the degree of liquidization in the technically important region of parameters.

It is surprizing that no attempt has been made so far to analyze thoroughly the nonlinear viscoelastic effects during the creeping flow regimes of PLF when, regardless of the rheological properties of the material, the oscillatory component of the stress may be expressed explicitly in the form

$$\tau_v = \varrho a \omega^2 (h - x) \sin(T) \quad (81)$$

so that the degree of liquidization during an arbitrary PLF equals the degree of liquidization under the corresponding simple periodic shearing flow with the amplitude of the shear $\varrho a \omega^2 h$ and the mean $\varrho g_z h$.

LIST OF SYMBOLS

| | |
|---|---|
| $a', a$ | amplitude of the wall oscillations |
| $A = \mathrm{Fe}\,\mathrm{Re}^{-1}$ | |
| $B = \mathrm{Fr}\,\mathrm{Re}^{-1/(1+n)}$ | |
| $C = \tau_0/\tau_s$ | |
| $\mathrm{Fr} = a\omega^2/g_z$ | |
| $g_z'$ | longitudinal component of gravitational acceleration |
| $g_z$ | effective steady-state mass acceleration |
| $h$ | hydraulic radius, film thickness |
| $h_0$ | film thickness free of vibrations |
| $H = h/L_B$ | |
| $K$ | coefficient of consistency |
| $L_B = (K\varrho^{-1}a^{-1+n}\omega^{-2+n})^{1/(1+n)}$ | |
| $M = \mathrm{Fr}^{1/n-1}\,\mathrm{Re}^{-1/n}$ | |
| $\mathcal{M}_T()$ | operator of time-averaging |
| $n$ | flow index |
| $p', p$ | isotropic pressure |
| $P_0', P_0$ | amplitude of oscillatory component of pressure drop |
| $P_s', P_s$ | steady-state component of pressure drop |
| $Q$ | intensity of irrigation |
| $\mathrm{Re} = \varrho h^{1+n} a^{1-n} \omega^{2-n} K^{-1}$ | |
| $S = [\partial_X V]^n = \tau K^{-1}(a\omega/h)^{-n}$ | |
| $t$ | time |
| $T = \omega t$ | |
| $v', v$ | longitudinal velocity |

| | |
|---|---|
| $v_m$ | mean velocity of flow |
| $V = v/(a\omega)$ | |
| $x', y', z'$ | cartesian coordinates in an inertial frame of reference |
| $x, y, z$ | cartesian coordinates in canonic frame of reference |
| $X = x/h$ | |
| $Y = x/L_B$ | |
| $\alpha_0, \alpha_s$ | parameters of asymptotic representation (*23*) |
| $\beta(n, C) = \mathscr{M}_T([1 + C \cos T]^{1/n})$ | |
| $\delta$ | thickness of the oscillatory boundary layer |
| $\varkappa(n)$ | function defined by Eq. (*74*) |
| $\lambda = \delta/L_B$ | |
| $\omega$ | frequency of oscillations |
| $\varrho$ | density |
| $\sigma = \tau_v/\tau_c$ | |
| $\tau = \tau_{xz}$ | shear stress |
| $\tau_s$ | local steady-state stress |
| $\tau_0$ | local amplitude of oscillatory stress |
| $\tau_v$ | field of oscillatory stress |
| $\tau_B = (K(\varrho a^2 \omega^3)^n)^{1/(1+n)}$ | |
| $\tau_c = \varrho a \omega^2 h$ | |
| $\phi(T)$ | local normalized oscillatory stress |
| $'$ | designation of quantities in an inertial frame of reference |
| $\overline{\phantom{x}}$ | operator of time-averaging |

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